Smooth-Rolling Knots

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Abstract

Morton's knots are a family of space curves to which there is no plane simultaneously tangent in three distinct points. This property enables a physical instance to roll on a plane while having at most two contact points with it. These knots are not smooth-rolling, i.e. they require a force to provoke a rolling motion, during which their center of mass oscillates up and down. By drawing a connection between Morton's knots and smooth-rolling Two-Disk Rollers, we design new smooth-rolling knots. These curves preserve the mesmerizing rolling behavior of Morton's knots while requiring only an infinitesimal force to be set in motion. We apply our method to a variety of knots beyond Morton's family, obtaining tightly-winding, smooth-rolling knots with different topologies.

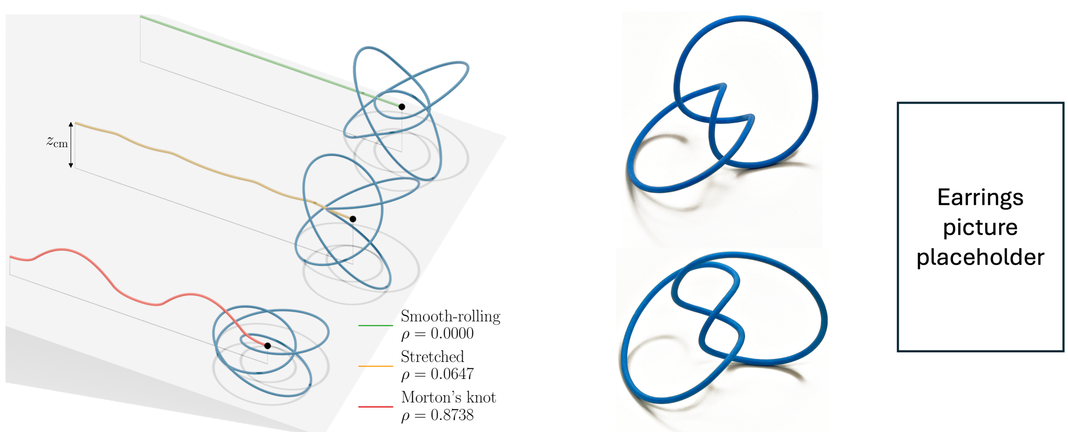

Figure 1: *Left: The center of mass heights z_{cm} along the rolling motion of knots with $a = 0.9$. Right: A 3D-printed (Multijet printing) optimized knot with $a = 0.7$ and $\rho = 0.0$.*

Morton's Knots and Smooth-Rolling Structures

Morton's knots [4] are a geometric family of trefoil knots defined by the following parametrization:

$$x_M(t) = \frac{ca\cos(3t)}{1 - b\sin(2t)}, \qquad y_M(t) = \frac{ca\sin(3t)}{1 - b\sin(2t)}, \qquad z_M(t) = z\frac{cb\cos(2t)}{1 - b\sin(2t)}, \qquad t \in [0, 2\pi] \tag{1}$$

where $a \in (0, 1)$ is a shape parameter, $a^2 + b^2 = 1$, $c = \frac{a}{1+b}$ is an isotropic scaling factor, and z encodes an anisotropic scaling factor along the z-axis. These knots have been studied by Eget and coauthors [1] for their rolling behavior. Knots in this class are *tritangentless* according to Morton's definition [4], i.e. there exists no plane tangent to the curve in three distinct points. Tritangentless knots can roll continuously on a plane without ever having more than two contact points with it.

To study the rolling smoothness, Eget et al. measure the range-to-average ratio, denoted by ρ:

$$\rho = \frac{\max z_{\mathrm{cm}} - \min z_{\mathrm{cm}}}{\overline{z}_{\mathrm{cm}}}, \tag{2}$$

where z_{cm} is the center of mass height and $\overline{z}_{\mathrm{cm}}$ its average over the rolling motion. The smaller the ρ measure, the smoother the motion. The smoothest-rolling objects have $\rho = 0$ and are said to be *smooth-rolling*, such as a right cylinder with a circular base. With an elliptical base, we get $\rho > 0$ due to vertical oscillations of the center of mass with periodic exchanges between kinetic and potential energy. For smooth-rolling objects, the onset of rolling motion can be achieved with an arbitrarily small force, or equivalently with an arbitrarily small tilt of the support plane. The center of mass height variation is visualized in Figure 1 for Morton's knot with $a = 0.9$.

Eget and colleagues look for the smoothest-rolling of Morton's knots by minimizing ρ with respect to a. To further reduce ρ for all values of a, they propose to optimize over the z scaling factor. Building upon these results, Dzojic and colleagues [3] apply additional scaling transformations to obtain even smaller values of ρ. However, none of these methods yield smooth-rolling knots.

We introduce a new method to obtain smooth-rolling knots by transforming Morton's knots. The method stems from the observation that the optimally stretched knots by Eget and colleagues have external lobes that visually resemble a Two-Disk Roller (TDR) [2].

TDRs are made of two identical orthogonal elliptical disks, characterized by α and β, the lengths of the ellipse half-axes and by the distance γ between the two ellipse centers (see Figure 2). The oloid, an iconic example of a rolling structure [5], can be seen as the convex hull of such disks with $\alpha = \beta = \gamma$, and both circles passing through each other's centers. As is the case for any generic TDR, the center of mass of the oloid oscillates up and down while rolling. Engelhardt and coauthors [2] identify a two-parameter class of smooth-rolling TDRs when they satisfy $\gamma^2 = 4\alpha^2 - 2\beta^2$.

We frame the search for smooth-rolling curves that approximate a given knot as an optimization problem solved in two stages. First fit a smooth-rolling TDR to the curve while allowing vertical stretching of the knot, then morph the curve within the convex hull of the TDR, controlling the trade-off between curve smoothness and knot approximation, all while preserving the smooth-rolling behavior. Our method is also applied to knots outside of Morton's family, leading to the design of new sets of smooth-rolling curves with various topologies.

Method

We want to find an optimal knot K^* that minimizes a distance measure d from a family of curves C subject to a smooth-rolling constraint. We are interested in the case where $C = K_a(z) = \{(x_M(t), y_M(t), z_M(t)) \in \mathbb{R}^3,\ t \in [0, 2\pi]\}$ are knots from family (1), where the shape parameter a is fixed and the scaling parameter z is free.

Assuming a homogeneous mass density along the curve, by symmetry, the center of mass of K_a is always located at the origin. In such symmetric knots, only the points lying on the convex hull of the curve affect the rolling behavior. We can therefore restrict the search for K^* to curves whose convex hull guarantees $\rho(K^*) = 0$, such as the ellipses of a smooth-rolling TDR.

Figure 2 illustrates the computation of a smooth-rolling curve in two stages. We first look for the TDR that best fits the exterior lobes of the target knot, i.e. the portions of the curve defining the convex hull. The vertical stretching proposed by Eget et al. aligns the exterior lobes to orthogonal ellipses (left vs. middle knots in Figure 2). We then fix the exterior to this smooth-rolling TDR, and morph the interior segments by minimizing a metric that strikes a balance between global curve smoothness and approximation of K_a.

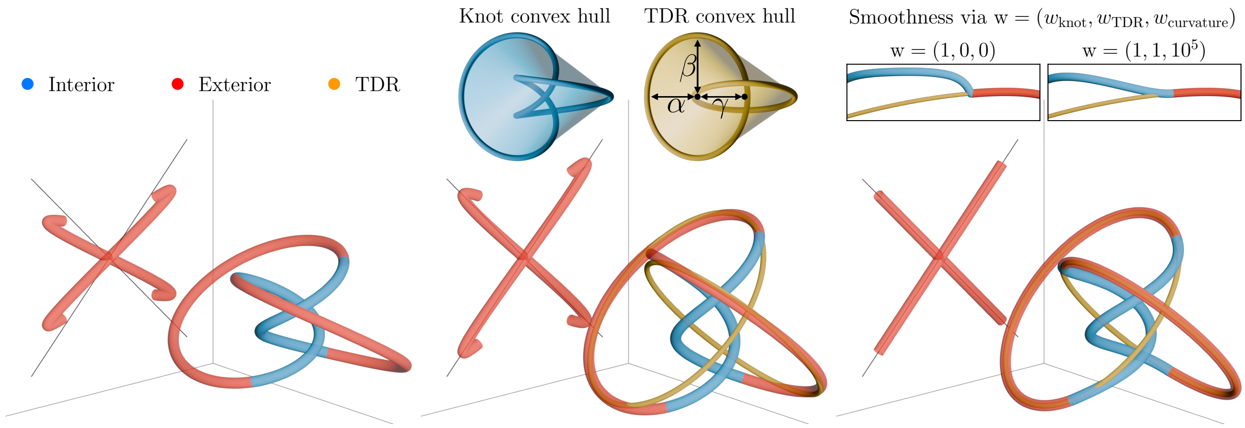

Figure 2: *Given Morton's knot K_a (left), we compute the z-stretch and the best-fitting TDR (middle), then morph the knot to match the convex hull of the TDR while preserving curve smoothness (right).*

Two-Disk Roller Fitting. Given a, we determine the optimal parameters α^*, β^*, z^* of the smooth-rolling TDR $T(\alpha, \beta)$ that best fits $K_a(z)$ by minimizing the distance $d(\mathcal{H}[K_a(z)], T(\alpha, \beta))$, where $\mathcal{H}[C]$ denotes the portions of curve C lying on its convex hull. Since affine transformations preserve convexity, a scaling of C results in a corresponding scaling of $\mathcal{H}[C]$. Therefore by pre-computing the convex hull, we circumvent the issue of differentiation through a potentially discontinuous operation and efficiently solve the optimization problem via gradient-based optimization. We discretize K_a and T, compute the mean-minimum distance d from the knot's hull to the TDR ellipses, and solve the distance minimization problem in Python via automatic differentiation and L-BFGS-B [7]. A result of this step is shown in Figure 2.

Knot Morphing. In the second optimization stage, we project $K_a(z^*)$ onto the convex hull of $T(\alpha^*, \beta^*)$. Using naive closest-point projection would create discontinuities and sharp kinks at the junction between the interior points of the knot and the TDR convex hull, as shown in Figure 2. We formulate a second optimization problem over the curve by introducing an interpolating cubic spline with approximately equispaced control points sampled from the result of the previous stage. The exterior control points are pinned in place while the interior ones, collectively named $\mathbf{q}$, are the variables of the following problem:

$$\text{minimize } w_{\text{knot}} E_{\text{knot}}(\mathbf{q}) + w_{\text{curvature}} E_{\text{curvature}}(\mathbf{q}) + w_{\text{TDR}} E_{\text{TDR}}(\mathbf{q}). \tag{3}$$

Here, E_{knot} penalizes the distance between the optimized knot and $K_a(z^*)$, while $E_{\text{curvature}}$ and E_{TDR} ensure curve smoothness by penalizing the discrete curvature at the interior-exterior junction and attracting interior points to the TDR. All quantities are computed by sampling the spline. Assuming $w_{\text{knot}} = 1$, the weights w_{TDR} and $w_{\text{curvature}}$ are free parameters to be set by the user that allow exploring the trade-off between curve smoothness and approximation quality (see Figure 2). The union of the TDR convex hull with the interior points of the optimized curve gives us a candidate smooth-rolling knot K^*. Since some points might be pushed outside of $\mathcal{H}[T]$ during the optimization process, we computationally check that $\rho(K^*) = 0$. If needed, we increase w_{TDR} until this condition is met.

Results

The first two columns of Figure 3 shows results of our algorithm for Morton's knots with $a = 0.3$ and 0.5. While the center of mass of Morton's knot and the stretched knot exhibit vertical oscillations ($\rho > 0$), it has constant height for all optimized knots. Besides Morton's knots, which are (3, 2)-torus knots, we also

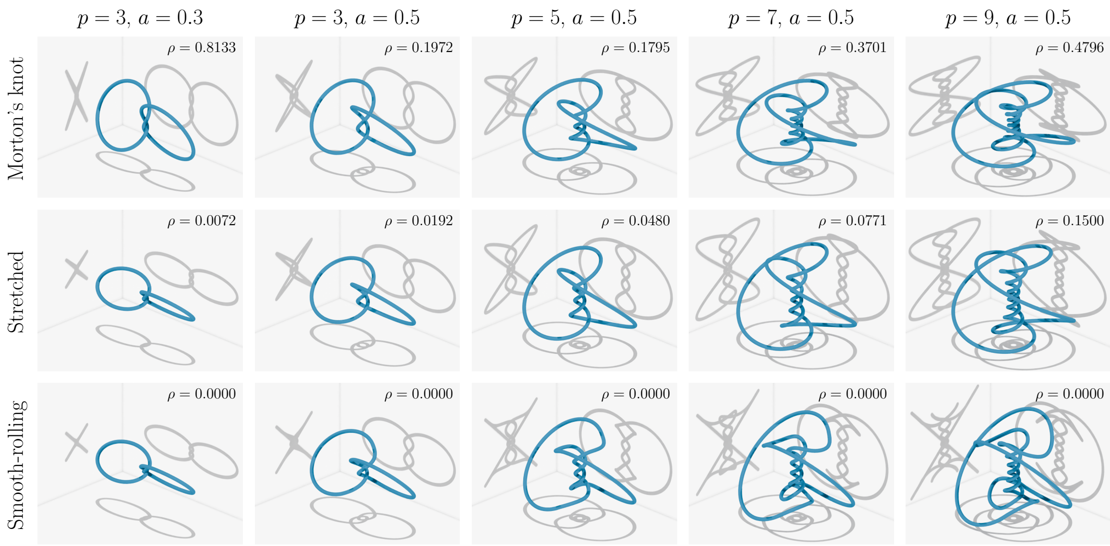

Figure 3: *Morton's knots, vertically stretched knots, and optimized knots for various a and p values.*

consider other $(p, 2)$-torus knots with odd integer p, as suggested by Eget et al. Even though they are not tritangentless anymore, these knots still roll. They feature two exterior lobes, analogous to TDRs, and have intriguing, tightly-winding curves in their interior. Figure 3 showcases our results for values of $p = 5, 7$ and 9. Our code, as well as videos and 3D models, can be found at go.epfl.ch/smooth_rolling_knots.

Conclusion

In this research, we achieve perfect rolling smoothness by transforming Morton's knots to match the convex hulls of smooth-rolling TDRs. An exciting next step could be to obtain smooth-rolling knots via TDR convex hulls without starting from Morton's knots. It would also be worthwhile to explore the general problem of finding such curves without the restriction of a TDR convex hull. A potential application is the combination of our results with trajectory optimization [6] for the design of mechanical systems or robotic devices, where rolling components with smooth and predictable movements are crucial. We also envision artistic applications of our methods, with smooth-rolling knots serving as the foundation for elegant kinetic sculptures.

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